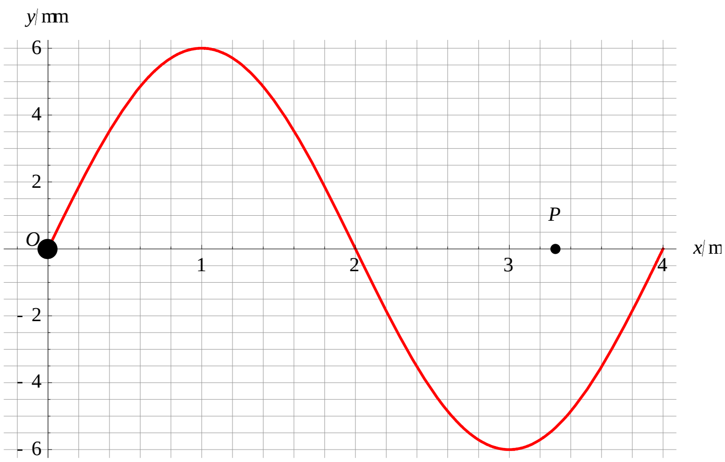
Teacher notes
Topic C

About phase difference.

This note discusses the idea of phase difference for travelling and standing waves.

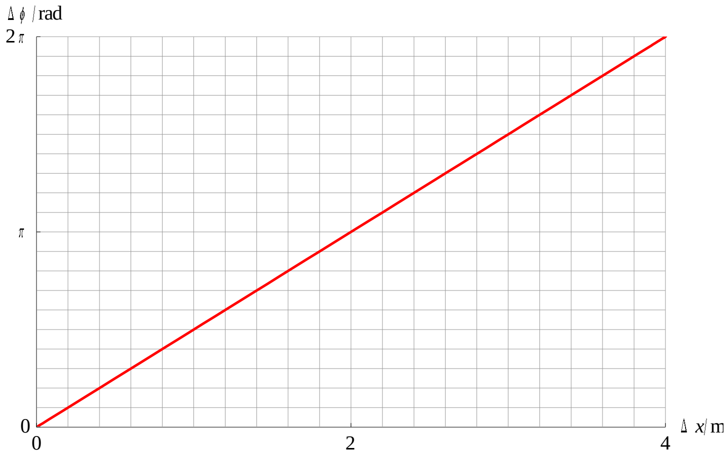
The graph shows, at t = 0, the shape of a string on which a transverse wave is travelling to the right. The wavelength is 4.0 m.



We may ask about the phase difference $\Delta\phi$ between an arbitrary point P in the medium and the point O at the origin. We use the formula:

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x$$

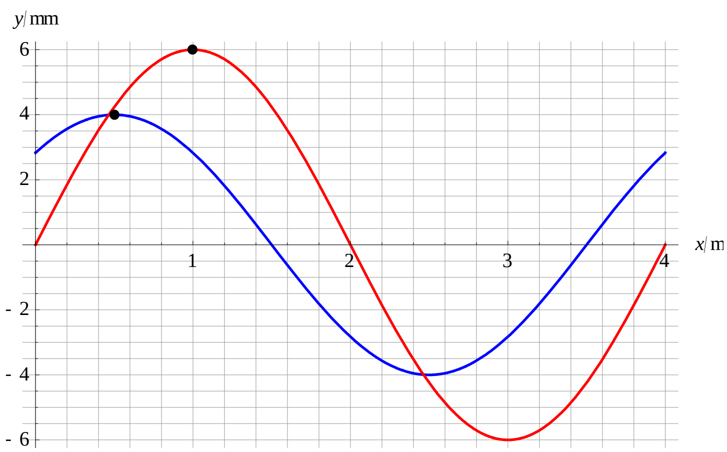
where Δx is the separation of points P and O. We see that the phase difference increases linearly with Δx starting from zero when P coincides with O and reaching the value 2π when P is a distance of one wavelength from O:



This is the situation for two points on the **same** travelling wave.

We may apply the same idea to two different travelling waves of the same wavelength and frequency and ask for the phase difference between them.

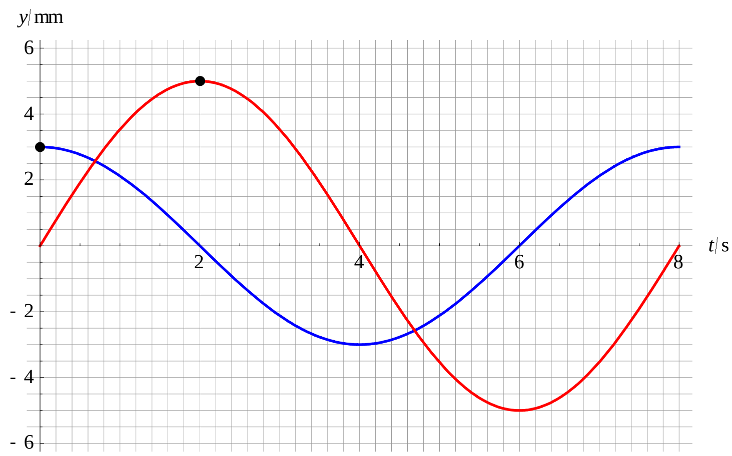
For example, we may have two different travelling waves that travel through the same medium. At the same time, say t = 0, the two waves look like:



The two peaks closest to each other differ by a distance 0.50 m. The phase difference between the two waves is, applying the same formula above,

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{4.0}\times 0.50 = \frac{\pi}{4}$$

We may also find the phase difference between two waves which arrive at the same point in space. The variation with time of the displacement of each wave at that same point is given by the graphs:



The two peaks closest to each other differ by a time 2.0 s. To find the phase difference between the two waves we need a variation of the formula above:

$$\Delta\phi = \frac{2\pi}{\lambda}\Delta x = \frac{2\pi}{\lambda}v\Delta t = \frac{2\pi}{T}\Delta t$$

where T is the period. So, in our example,

$$\Delta\phi = \frac{2\pi}{T}\Delta t = \frac{2\pi}{8.0}\times 2.0 = \frac{\pi}{2}$$

In two-source interference we are interested in the path difference at the point P of observation from each of the sources. Let us see the connection of the path difference with phase difference. From the formula above $\Delta\phi = \frac{2\pi}{T}\Delta t$ where Δt now means the extra time taken by the wave from one source to get to P. This extra time is given by $\Delta t = \frac{\text{path difference}}{v}$ and hence

$$\Delta\phi = \frac{2\pi}{T}\Delta t = \frac{2\pi}{T}\frac{\text{path difference}}{v} = \frac{2\pi}{\lambda}\times\text{path difference}$$

And connects nicely with the original formula $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$.

The discussion above applies to travelling waves **only**.

A standing wave is the superposition of two identical waves travelling in opposite directions. The equation of such a wave is therefore fundamentally different from that of a travelling wave. It is

$$y = y_{\mathrm{R}} + y_{\mathrm{L}}$$

where $y_R = A\sin(\frac{2\pi}{\lambda}x - \omega t)$ and $y_L = A\sin(\frac{2\pi}{\lambda}x + \omega t)$. Then

$$\begin{aligned} y &= A\sin(\frac{2\pi}{\lambda}x - \omega t) + A\sin(\frac{2\pi}{\lambda}x + \omega t) \\ &= 2A\sin(\frac{2\pi}{\lambda}x)\cos(\omega t) \end{aligned}$$

This shows that all points oscillate with the same frequency, but they have different amplitude. The amplitude is $2A\sin(\frac{2\pi}{\lambda}x)$ and depends on the position of the point of interest.

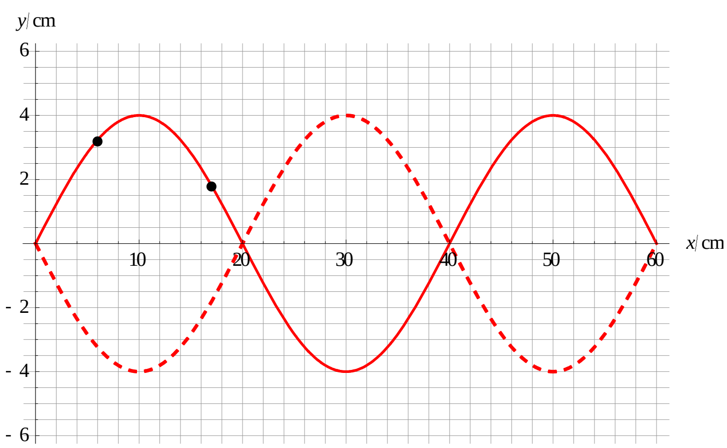
Suppose for convenience we look at a standing wave on a string of length *L* with both ends fixed. The displacement is given by $y = 2A\sin(\frac{2\pi}{\lambda}x)\cos(\omega t)$. At both ends (*x* = 0 and *x* = *L*) we have nodes and so *y* = 0 all the time. The displacement equation ensures this automatically for the end *x* = 0 but we have to impose *y* = 0 at *x* = *L*. This means that

$$\begin{aligned} &\sin(\frac{2\pi L}{\lambda}) = 0 \Rightarrow \frac{2\pi L}{\lambda} = n\pi \\ &\Rightarrow \lambda = \frac{2L}{n},\ n = 1,2,3,\ldots \end{aligned}$$

This is how we get the condition relating the wavelength to the length of the string. Suppose that we have the third harmonic, i.e. *n* = 3. Then

$$\begin{aligned} y &= 2A\sin(\frac{2\pi}{\lambda}x)\cos(\omega t) \\ &= 2A\sin(\frac{3\pi}{L}x)\cos(\omega t) \end{aligned}$$

In addition to the nodes at the ends we have nodes at $x = \frac{L}{3}$ and $x = \frac{2L}{3}$. This is shown for the case *L* = 60 cm.



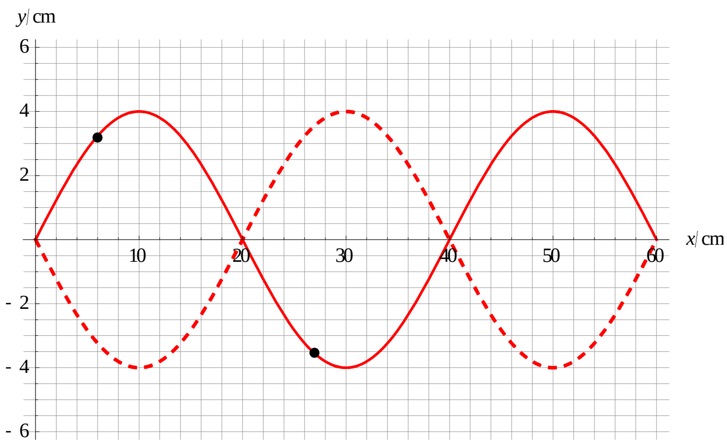
This is the conventional way of representing a standing wave: the solid line shows the wave at t = 0 and the dashed line half a period later.

What is the phase difference between the two marked points?

The truth is that we do not have a satisfactory definition of phase difference for standing waves in this course! Naively, we say that at this instant both points move in the same direction (downward) and so we assign zero phase difference between them. (The formula $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$ does not give zero phase difference.) Since the displacement equation is $y = 2A\sin(\frac{3\pi}{L}x)\cos(\omega t)$ we look at the term with time ($\cos(\omega t)$) and see that nothing is added to ωt. By analogy with what we did in SHM we would say that each point has zero phase and therefore zero phase difference. So, all points within the loop from x = 0 to x = 20 cm have zero phase difference between them. The same applies to any two points within the same loop such as for example points within [20 cm, 40 cm] and [40 cm, 60 cm].

I.e. points within the same loop have zero phase difference between them.

What about points within consecutive loops such as those shown below?



The naïve argument would say that the point in the first loop is moving down and the other point is moving up so we should assign a phase difference of π. (Again, the formula $\Delta\phi = \frac{2\pi}{\lambda}\Delta x$ does not give π.) The first point has displacement $y = 2A\sin(\frac{3\pi}{L}x_1)\cos(\omega t)$ and the second $y = 2A\sin(\frac{3\pi}{L}x_2)\cos(\omega t)$.

Notice that the term is $2A\sin(\frac{3\pi}{L}x_1)$ is the amplitude of the first point and is a positive quantity. So we are justified in assigning zero phase to the first point since nothing is added to ωt. For the second point the quantity $2A\sin(\frac{3\pi}{L}x_2)$ is negative. It is the negative of the amplitude. So, for this point we can write

$$y = 2A\sin(\frac{3\pi}{L}x_2)\cos(\omega t)$$
$$= -2A\sin(\frac{3\pi}{L}x_2)\cos(\omega t + \pi)$$

The term in front of the cosine is the positive amplitude and we can now see what has been added to ωt . It is π and so the second point has phase π. This means the phase difference between the two points in consecutive loops is π.

So, for standing waves, the only possible values of phase difference between any two points are 0 and π. It is zero for points within the same loop and π for points in consecutive loops.

The graph shows the phase difference between the point at the left end of the string and any other point. This graph is to be contrasted with the equivalent graph for a travelling wave shown earlier.